The computer program Big Deal uses a one-to-one correspondence between numbers and bridge-deals, i.e. each possible bridge-deal receives its own 'personal identification' number. It uses a random number generator to produce such an identification number, which is then converted into a bridge-deal. The conversion from a given number to its corresponding bridge-deal is handled by the procedure 'code-to-hand' in the program. This note will explain the mechanism of this procedure.

Let G denote the total number of possible bridge-deals (one can show that $G = \binom{52}{13}\binom{39}{13}\binom{26}{13}$). There are of course numerous ways to establish a one-to-one correspondence between all possible bridge-deals and the numbers 0 to $G-1$, so we will start explaining the particular choice that was made for Big Deal.

First, numbers are assigned to the cards. This was done as follows: Ace of Spades gets number 1, King of Spades is number 2, and so on. After the Spades are numbered, it is the turn of the Hearts, then Diamonds and finally Clubs. Two of Clubs is therefore the last card in the sequence and it has number 52.

Let us now look at a possible hand for North. Reading the card numbers from low to high, we see that this hand corresponds to a strictly increasing sequence of 13 numbers, where the numbers are taken from the set $\{1, 2, \ldots, 52\}$. For mathematical convenience, we wish to make a slight modification to this observation: if, in this strictly increasing sequence, we subtract i from the i-th number, we obtain a non-decreasing sequence of 13 numbers, where the numbers are taken from the set $\{0, 1, \ldots, 39\}$. We conclude that there is a one-to-one correspondence between the hands for North and the non-decreasing number sequences of length 13, where the numbers are taken from $\{0, 1, \ldots, 39\}$.

Now, knowing North's hand, we turn to East and we wonder if we can find a one-to-one correspondence between the possible hands for East and number sequences, similar to the way we did for North. Of course, one can still argue that East's hand can be recovered from a non-decreasing number sequences of length 13, with numbers taken from $\{0, 1, \ldots, 39\}$. This is true, but we do not obtain a one-to-one correspondence in this way. To this end, number the remaining cards (the cards that are not in North's hand) from 1 to 39. We see that the cards in East's hand correspond to a strictly increasing number sequence of length 13, where the numbers are taken from the set $\{1, 2, \ldots, 39\}$. Again, we prefer to have a slightly modified representation. In this strictly increasing sequence we subtract i from the i-th number, thus obtaining a non-decreasing sequence of 13 numbers, where the numbers are taken from the set $\{0, 1, \ldots, 26\}$. We conclude that there is a one-to-one correspondence between the hands for North and the non-decreasing number sequences of length 13, where the numbers are taken from $\{0, 1, \ldots, 26\}$.

Knowing both North's and East's hand, there is a one-to-one correspondence between the possible hands for South and the non-decreasing number sequences of length 13, where the numbers are taken from $\{0, 1, \ldots, 13\}$. Finally, if the hands of North, East and South are known, then also West's hand is known, so no coding is needed for West.

So, we can represent any bridge-deal in a unique way by a non-decreasing number sequence of length 13 with numbers from $\{0, \ldots, 39\}$, followed by a non-decreasing sequence of length 13 with numbers from $\{0, \ldots, 26\}$, terminated by a non-decreasing sequence of length 13 with numbers from $\{0, \ldots, 13\}$. Let us refer to such a sequence as a *bridge-sequence*. The representation of bridge-deals by bridge-sequences suggests the following ordering of the possible bridge-deals. The bridge-deal corresponding to the sequence consisting of 39 zeros is first in this ordering. The sequence consisting of 13 times 39, followed by 13 times 26, and terminated by 13 times 13 is last in this ordering. For any bridge-sequence, unequal to the last one, say $a_0, \ldots, a_{12}, b_0, \ldots, b_{12}, c_0, \ldots, c_{12}$, we construct its *successor* as follows. Search for the right-most number in the sequence that is not at its maximum (i.e. less than 39 for one of the numbers a_i, less than 26 for one of the number b_i or less than 13 for one of the numbers c_i). Let us denote this element of the sequence by x_ℓ, where x stands for one the letters a, b, c and $\ell \in \{0, \ldots, 12\}$. The number x_ℓ is increased by 1. The numbers left from it remain unchanged, and the numbers right from it are set to the lowest possible value that still results in a valid bridge-sequence: the number y_k is set to $x_\ell + 1$ if the letter y (seen as a variable taken from the set $\{a, b, c\}$) equals the letter x, and if $k > \ell$. It is set to zero if the letter y is lexicographically greater than x. This works much like a kilometer counter. The differences with a normal kilometer counter are only to ensure that the successor of any bridge-sequence is again a valid bridge-sequence. Starting at the zero-sequence and incrementing in the way prescribed above, our kilometer counter will reach any given bridge-sequence. It is now clear how we intend to assign numbers to the bridge-sequences. The zero-sequence is assigned 0, its successor is assigned 1, etc. Let us call this number the Gödel number of a bridge-sequence.

It turns out that an explicit formula can be given for the Gödel number of a bridge-sequence. For the bridge-sequence $s := (a_0, \ldots, a_{12}, b_0, \ldots, b_{12}, c_0, \ldots, c_{12})$, define

$$G(s) = \binom{39}{13}\binom{26}{13}G^a(s) + \binom{26}{13}G^b(s) + G^c(s),$$

where

$$G^a(s) = \sum_{j=0}^{12}(\binom{52 - a_{j-1} - j}{13 - j} - \binom{52 - a_j - j}{13 - j}),$$

$$G^b(s) = \sum_{j=0}^{12}(\binom{39 - b_{j-1} - j}{13 - j} - \binom{39 - b_j - j}{13 - j}),$$

and

$$G^c(s) = \sum_{j=0}^{12}(\binom{26 - c_{j-1} - j}{13 - j} - \binom{26 - c_j - j}{13 - j}).$$

In these formulae we should read $a_{-1} = 0$, $b_{-1} = 0$ and $c_{-1} = 0$. We claim that $G(s)$ is the Gödel number of s. This claim is trivially true for the zero-sequence. Now it suffices to prove that $G(s') = G(s) + 1$ whenever s' is the successor of s.

Let x_ℓ denote the right-most number in the sequence s that is not maximal. Let us assume for the moment that the variable letter x stands for the letter a. Then $G^b(s') = G^c(s') = 0$, since the last 26 numbers in the s'-sequence are 0. Further, we have $G^b(s) = \binom{39}{13} - 1$ and $G^c(s) = \binom{26}{13} - 1$, since the last 26 numbers in the s-sequence are all maximal. Finally, note that

$$\begin{aligned} G^a(s') - G^a(s) &= -\binom{52 - a_{\ell-1} - \ell}{13 - \ell} + \binom{52 - a_\ell - \ell}{13 - \ell} \\ &\quad -\binom{52 - a_{\ell-2} - \ell + 1}{13 - \ell + 1} + \binom{52 - a_{\ell-1} - \ell + 1}{13 - \ell + 1} \\ &\quad +\binom{52 - a_{\ell-2} - \ell + 1}{13 - \ell + 1} - \binom{52 - a_{\ell-1} - \ell}{13 - \ell + 1} \\ &= 1. \end{aligned}$$

We conclude that indeed

$$G(s') = G(s) + \binom{39}{13}\binom{26}{13} - \binom{26}{13}(\binom{39}{13} - 1) - (\binom{26}{13} - 1) = G(s) + 1.$$

The proof that $G(s') = G(s) + 1$ is similar when the right-most element that is not maximal is positioned in the b-block or the c-block, instead of the a-block.

The procedure 'code-to-hand' in the program Big Deal converts a given Gödel number to its associated bridge-sequence. The pseudo-code for computing the first 13 elements in the bridge-sequence is given below.

Input: A (Gödel) number $g \in \{0, 1, \ldots, G-1\}$.

$a_{-1} := 0;$
$g_{-1} := g;$
for $j := 0$ **to** 12 **do**
begin
$\quad a_j := \max\{a \mid \binom{39}{13}\binom{26}{13}(\binom{52-a_{j-1}-j}{13-j} - \binom{52-a-j}{13-j}) \leq g_{j-1}\};$
$\quad x_j := \binom{39}{13}\binom{26}{13}(\binom{52-a_{j-1}-j}{13-j} - \binom{52-a_j-j}{13-j});$
$\quad g_j := g_{j-1} - x_j;$
end;

In order to show that this code correctly computes the first 13 elements of the bridge-sequence associated with g, we will prove by induction that

$$0 \leq a_{j-1} \leq a_j \leq 39$$

for all $j \in \{0, \ldots, 12\}$. We will also prove that

$$g_j + \sum_{i=0}^{j} x_i = g$$

for all $j \in \{-1, 0, \ldots, 12\}$, and that

$$0 \leq g_j < \binom{39}{13}\binom{26}{13}(\binom{52 - a_j - j}{13 - j} - \binom{52 - a_j - 1 - j}{13 - j})$$

for all $j \in \{-1, 0, \ldots, 12\}$.

For $j = -1$, the second claim is trivially true. For $j = -1$, the third claim states that $g_{-1} < \binom{39}{13}\binom{26}{13}(\binom{53}{14} - \binom{52}{14})$. Note that $\binom{53}{14} - \binom{52}{14} = \binom{52}{13}$, hence this claim is equivalent to $g_{-1} < G$, which is also trivially true.

Now, let $j \geq 0$ and assume that the second and third claim are true for $j-1$. We will show that all three claims are true for j. Note that the inequality

$$\binom{39}{13}\binom{26}{13}(\binom{52 - a_{j-1} - j}{13 - j} - \binom{52 - a - j)}{13 - j}) \leq g^{j-1}$$

is satisfied for $a = a_{j-1}$, hence $a_j \geq a_{j-1}$. From the fact that the third claim is true for $j-1$, it follows that this equality is violated for all $a > 39$. Hence, the number a_j is well-defined, and it satisfies $0 \leq a_{j-1} \leq a_j \leq 39$, i.e. the first claim is true for j.

The proof that the second claim is true for j follows trivially by combining $g_j = g_{j-1} - x_j$ and $g_{j-1} + \sum_{i=0}^{j-1} x_i = g$.

It follows from the definition of a_j that

$$\binom{39}{13}\binom{26}{13}(\binom{52 - a_{j-1} - j}{13 - j} - \binom{52 - a_j - 1 - j)}{13 - j}) > g_{j-1}.$$

Using this inequality we obtain

$$g_j = g_{j-1} - x_j < \binom{39}{13}\binom{26}{13}(\binom{52 - a_j - j}{13 - j} - \binom{52 - a_j - 1 - j}{13 - j}),$$

and we see that also the third claim is true for j.

The computation of the next 13 elements in the bridge-sequence can be done by similar code. The input-number for this second part is the value g_{12} computed in the first part. Similarly, one can show that this will produce a non-decreasing sequence $b_0, \ldots, b_{12}$ with elements from $\{0, \ldots, 26\}$, Numbers $g_{13}, \ldots, g_{25}$ and $x_{13}, \ldots, x_{25}$ will be generated satisfying

$$0 \leq g_{j+13} < \binom{26}{13}(\binom{39-b_j-j}{13-j} - \binom{39-b_j-1-j}{13-j})$$

for all $j \in \{-1, 0, \ldots, 12\}$, and

$$g_j + \sum_{i=0}^{j} x_i = g$$

for all $j \in \{-1, 0, \ldots, 25\}$.

Finally, in the third part a non-decreasing sequence $c_0, \ldots, c_{12}$ with elements from $\{0, \ldots, 13\}$ is computed together with numbers $g_{26}, \ldots, g_{38}$ and $x_{26}, \ldots, x_{38}$ satisfying

$$0 \leq g_{j+26} < (\binom{26-c_j-j}{13-j} - \binom{26-c_j-1-j}{13-j})$$

for all $j \in \{-1, 0, \ldots, 12\}$, and

$$g_j + \sum_{i=0}^{j} x_i = g$$

for all $j \in \{-1, 0, \ldots, 38\}$.

The outcome of the whole procedure is therefore a valid bridge-sequence, say s. The Gödel number of this bridge-sequence is easily seen to satisfy

$$G(s) = \sum_{i=0}^{38} x_i.$$

Applying the result

$$g_j + \sum_{i=0}^{j} x_i = g$$

for $j = 38$, we obtain

$$G(s) + g_{38} = g.$$

Applying the result

$$0 \leq g_{j+26} < (\binom{26 - c_j - j}{13 - j} - \binom{26 - c_j - 1 - j}{13 - j})$$

for $j = 12$, we obtain

$$0 \leq g_{38} < \binom{14 - c_{12}}{1} - \binom{13 - c_{12}}{1} = 1,$$

hence $g_{38} = 0$. Therefore $G(s) = g$, and we see that we have actually computed the (unique) bridge-sequence with Gödel number g.